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TECHNICAL MEMORANDUM

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THE PROBLEM OF TORSION IN PRISMATIC MEMBERS OF CIRCULAR SEGMENTAL CROSS SECTION

By A. Weigand

TRANSLATION

"Das Torsionsproblem für Stäbe von kreisabschnittförmigem Querschnitt"

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THE PROBLEM OF TORSION IN PRISMATIC MEMBERS OF CIRCULAR SEGMENTAL CROSS SECTION*

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SUMMARY

The problem is solved by approximation, by setting up a function complying with the differential equation of the stress function, and determining the coefficients appearing in it in such a way that the boundary condition is fulfilled as nearly as possible.

For the semicircle, for which the solution is known, the method yields very accurate values; the approximated stress distribution is in good agreement with the accurately computed distribution. Stress and strain measurements indicate that the approximate solution is in sufficiently exact agreement with reality for segmental cross sections.

I. FUNDAMENTAL EQUATION OF TORSION AND ITS APPROXIMATE SOLUTION BY THE METHOD OF LEAST SQUARES

The torsion problem for the prismatic member stressed by twisting moments at the ends is formulated as follows. Find a function $f(y, z)$ which in the cross-sectional plane satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -1 \tag{1}$$

and at the boundary of the cross section the condition

$$\bar{f} = 0 \tag{2}$$

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*"Das Torsionsproblem für Stäbe von kreisabschnittförmigem Querschnitt." Luftfahrt-Forschung, Band 20, Lfg. 12, Feb. 8, 1944, pp. 333-340.

This function $f(y, z)$ then gives the torsion constant J_d of the member according to

$$J_d = 4 \int\int f(y, z) dy\, dz \tag{3}$$

the double integral to be extended over the cross section. The angle of twist ψ of a length l is

$$\psi = \frac{M_d l}{G J_d} \tag{4}$$

M_d the applied torque, G the modulus of rigidity of the material.

The components of the shearing stress follow from

$$\tau_{xy} = -\frac{2M_d}{J_d}\frac{\partial f}{\partial z}, \quad \tau_{xz} = \frac{2M_d}{J_d}\frac{\partial f}{\partial y} \tag{5}$$

Owing to the equations(5) which satisfy identically the equilibrium condition

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$f(y, z)$ is called the stress function of the torsion problem.

The differential equation (1) with the boundary condition equation (2) follows from the consideration of the state of strain and the relation between stress and strain, which is given by Hooke's law.

Occasionally, it is appropriate to introduce the polar coordinates r, φ instead of the rectangular coordinates (y, z) (fig. 1). The differential equation of the stress function together with the boundary condition then reads

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} = -1 \tag{1a}$$

$$\overline{f} = 0 \tag{2a}$$

while the torsion constant follows from

$$J_d = 4 \int \int f(r, \varphi) r \, dr \, d\varphi \tag{3a}$$

and the shearing stress components from

$$\left.\begin{aligned} \tau_r &= \frac{2M_d}{J_d} \frac{1}{r} \frac{\partial f}{\partial \varphi} \\ \tau_\varphi &= - \frac{2M_d}{J_d} \frac{\partial f}{\partial r} \end{aligned}\right\} \tag{5a}$$

Rigorous methods for solving the potential problem posed by equations (1) and (2) will not be discussed.

The approximate solution can be effected in three ways. A function can be assumed that satisfies equation (1) but not equation (2). If the differential equation is replaced by a variation problem, it results in the conventional Ritz method; or a function satisfying the differential equation can be assumed and the boundary condition met in individual points or "on the average;" an exact explanation of what is meant by "on the average" will be given later. Lastly the differential equation can be replaced by a difference equation and the linear equation system ensuing from the boundary condition solved by iteration with the aid of the Liebmann-Wolf method. Only the second method is discussed in the present report, after having been pointed out, among others, by Trefftz (reference 1) and St. Bergmann (reference 2).

Since the torsion problem of the segment is to be treated, we proceed from the differential equation (1a). It has the particular solutions

$$f_o = -\frac{r^2}{4}, \quad f_k = r^k \cos k\varphi, \quad g_k = r^k \sin k\varphi \tag{6}$$

from which the general solution

$$f = -\frac{r^2}{4} + \sum_o^n \left[a_k f_k + b_k g_k\right] \tag{7}$$

can be built up. Now the determination of the coefficients a_k and b_k is involved. The next thing is to so determine them that equation (2a) is complied within individual points. Among others, problems relating to plate bending have already been solved by this method.

Another way is the following: Rather than specifying strict compliance with the boundary condition at originally established points it is required that by choice of the coefficients the integral of the squares of the boundary values is least. In this instance the boundary condition is said to be fulfilled "on the average."

This method is hereafter called the method of least squares. In the formula the requirement on the factors reads

$$\bar{J} = \oint \bar{f}^2 \, \bar{ds} = \text{Min.} \quad \bar{ds} = \text{Boundary element} \tag{8}$$

The integral is to be extended over the entire boundary; this is indicated by the sign $\oint$. Putting equation (7) in equation (8) gives

$$\oint \left\{ -\frac{\bar{r}^2}{4} + a_o + \sum_1^n \left[a_k f_k(\bar{r}, \varphi) + b_k g_k(\bar{r}, \varphi) \right] \right\}^2 \bar{ds} = \text{Min.} \tag{9}$$

The coefficients follow from the requirement

$$\frac{\partial \bar{J}}{\partial a_o} = 0, \quad \frac{\partial \bar{J}}{\partial a_k} = 0, \quad \frac{\partial \bar{J}}{\partial b_k} = 0, \quad k = 1 \ldots n \tag{10}$$

From equation (10) follows a linear equation system for the $2n + 1$ unknown a_o, a_k, and b_k.

The practical use of the method depends upon whether sufficiently exact results consistent with a moderate amount of paper work are obtainable, especially for the stresses, or in other words without having to solve a great number of linear equations.

II. APPLICATION TO THE SEMICIRCLE AND THE SEGMENT

1. Semicircle; Strict and Approximate Solution by the Method of Least Squares

The strict solution of the torsion problem for the sector was given by St. Venant (Handb. d. Physik Bd VI, pp. 153-154). The special case of the semicircle is easily treated as will be shown.

To remain in agreement with the notation for the segment (fig. 6) the coordinate system of figure 2 is shown for the semicircle.

On the straight boundary AB, $\varphi = \frac{\pi}{2}$ and $3\frac{\pi}{2}$; in the cross section, $\frac{\pi}{2} \leqq \varphi \leqq \frac{3\pi}{2}$.

The stress function is expressed by

$$f(r,\ \varphi) = \sum_{1,3\ldots}^{\infty} X_k(r)\ \cos\ k\varphi \tag{11}$$

It already fulfills the boundary condition on the straight boundary, since k is an odd number. The constant 1 in the internal $\frac{\pi}{2} \leqq \varphi \leqq \frac{3\pi}{2}$ is expanded in a Fourier series.

$$1 = \frac{4}{\pi} \sum_{1,3\ldots}^{\infty} (-1)^{\frac{k+1}{2}} \frac{\cos\ k\varphi}{k} \tag{12}$$

Introducing equations (11) and (12) in equation (1a), the comparison of the coefficients of $\cos\ k\varphi$ on both sides of the equation gives

$$X_k'' + \frac{1}{r}X_k' - \frac{k^2}{r^2}X_k = -\frac{4}{\pi}\ \frac{(-1)^{\frac{k+1}{2}}}{k} \tag{13}$$

The solution of this differential equation, finite for $r = 0$, reads

$$X_k = C_k r^k - \frac{4}{k\pi}\ \frac{(-1)^{\frac{k+1}{2}}}{4 - k^2}\ r^2 \tag{14}$$

Since $X_k(R)$ must be = 0

$$C_k = \frac{4}{k\pi} \frac{(-1)^{\frac{k+1}{2}}}{4 - K^2} R^{2-k} \tag{15}$$

and, hence,

$$f(r, \varphi) = \frac{4R^2}{\pi} \sum_{1,3,5..}^{\infty} \frac{(-1)^{\frac{k+1}{1}}}{k(4 - k^2)} \left[\left(\frac{r}{R}\right)^k - \left(\frac{r}{R}\right)^2\right] \cos k\varphi \tag{16}$$

is the solution of the torsion problem for the semicircle. The torsion constant J_d and the shearing stress distribution are computed from equation (16). From equation (3a) follows on three places exactly

$$J_d = 0.297\, R^4 = \kappa R^4$$

and from (5a)

$$\tau_r = \frac{8}{\kappa\pi} \frac{M_d}{R^3} \left[\frac{1}{3}\left(1 - \frac{r}{R}\right) \sin \varphi + \frac{1}{3^2 - 4}\left(\frac{r^2}{R^2} - \frac{r}{R}\right) \sin 3\varphi \right.$$

$$\left. - \frac{1}{5^2 - 4}\left(\frac{r^4}{R^4} - \frac{r}{R}\right) \sin 5\varphi + - \ldots \right] \tag{17}$$

$$\tau_\varphi = \frac{8}{\kappa\pi} \frac{M_d}{R^3} \left[\frac{1}{3}\left(1 - 2\frac{r}{R}\right) \cos \varphi \right.$$

$$+ \frac{1}{3(3^2 - 4)}\left(3\frac{r^2}{R^2} - 2\frac{r}{R}\right) \cos 3\varphi$$

$$\left. - \frac{1}{5(5^2 - 4)}\left(5\frac{r^4}{R^4} - 2\frac{r}{R}\right) \cos 5\varphi + - \ldots \right] \tag{18}$$

The maximum shearing stress occurs at A (fig. 2), that is, for $r = 0$ and $\varphi = \frac{\pi}{2}$. Here

$$\tau_{max} = \frac{8}{3\kappa\pi} \frac{M_d}{R^3} = 2.85 \frac{M_d}{R^3} \tag{19}$$

The shearing stress at C (fig. 2) is

$$\tau_{\varphi}{}^{C} = 2.44 \frac{M_d}{R^3}$$

Following the rigorous solution for the semicircle an approximate solution by the method of least squares shall be derived.

Since the cross section is symmetrical about $\varphi = \pi$, and following equation (7) we write:

$$f = -\frac{r^2}{4} + \sum_{0}^{\infty} a_k r^k \cos k\varphi \tag{20}$$

Instead of coefficient a_k the quantity x_k is introduced by

$$ak = \frac{R^2}{4} \frac{x_k}{R^k} \tag{21}$$

So with $\lambda = \frac{r}{R}$, formula (20) reads

$$f = \frac{R^2}{4}\left(-\lambda^2 + \sum_{0}^{n} x_k \lambda^k \cos k\varphi\right) \tag{20a}$$

The boundary values are

$$\text{On AB (fig. 2)} \quad \bar{f}_{AB} = \frac{R^2}{4}\left(-\lambda^2 + \sum_{0}^{n} x_k \lambda^k \cos \frac{k\pi}{2}\right)$$

$$\text{On BC (fig. 2)} \quad \bar{f}_{BC} = \frac{R^2}{4}\left(-1 + \sum_{0}^{n} x_k \cos k\varphi\right)$$

The method of least squares yields as conditional equation for x_k

$$\int_0^1 \left(-\lambda^2 + \sum_0^n x_k \lambda^k \cos\frac{k\pi}{2}\right)^2 d\lambda + \int_{\frac{\pi}{2}}^{\pi} \left(-1 + \sum_0^n x_k \cos k\varphi\right)^2 d\varphi = \text{Min.} \quad (22)$$

Therefore

$$\int_0^1 \left(-\lambda^2 + \sum_0^n x_k \lambda^k \cos\frac{k\pi}{2}\right)\lambda^l \cos\frac{l\pi}{2}\, d\lambda$$

$$+ \int_{\frac{\pi}{2}}^{\pi} \left(-1 + \sum_0^n x_k \cos k\varphi\right) \cos l\varphi\, d\varphi = 0$$

For x_k the linear equation system with symmetrical matrix

$$\sum_0^n A_{kl}\, x_k = B_l \quad l = 0,1, \ldots \quad (23)$$

is applicable, with

$$A_{00} = 1 + \frac{\pi}{2} \quad (24a)$$

$$A_{kk} = \frac{\pi}{4} + \frac{\cos^2\frac{k\pi}{2}}{2k + 1} \quad k = 1, \ldots n \quad (24b)$$

$$A_{kl} = \frac{\cos\frac{k\pi}{2}\cos\frac{l\pi}{2}}{k + l + 1} - \frac{1}{2}\left[\frac{\sin\,(k - l)\frac{\pi}{2}}{k - l} + \frac{\sin\,(k + l)\frac{\pi}{2}}{k + l}\right] \quad k \neq l \quad (24c)$$

$$B_0 = \frac{1}{3} + \frac{\pi}{2};\; B_l = \frac{\cos\frac{l\pi}{2}}{l + 3} - \frac{\sin\frac{l\pi}{2}}{l} \quad l = 1, \ldots n \quad (24d)$$

The numerical calculation was effected for $n = 1, 2 \ldots 6$.
For $n = 6$ the equation system reads

$$
\begin{aligned}
&\left(1+\frac{\pi}{2}\right)x_0 - x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{5}x_4 - \frac{1}{5}x_5 - \frac{1}{7}x_6 = \frac{1}{3} + \frac{\pi}{2}\\
&- x_0 + \frac{\pi}{4}x_1 - \frac{1}{3}x_2 + \frac{1}{15}x_4 - \frac{1}{35}x_6 = -1\\
&- \frac{1}{3}x_0 - \frac{1}{3}x_1 + \left(\frac{1}{5}+\frac{\pi}{4}\right)x_2 - \frac{3}{5}x_3 - \frac{1}{7}x_4 + \frac{5}{21}x_5 + \frac{1}{9}x_6 = -\frac{1}{5}\\
&\frac{1}{3}x_0 - \frac{3}{5}x_2 + \frac{\pi}{4}x_3 - \frac{3}{7}x_4 + \frac{1}{9}x_6 = \frac{1}{3}\\
&\frac{1}{5}x_0 + \frac{1}{15}x_1 - \frac{1}{7}x_2 - \frac{3}{7}x_3 + \left(\frac{1}{9}+\frac{\pi}{4}\right)x_4 - \frac{5}{9}x_5 - \frac{1}{11}x_6 = \frac{1}{7}\\
&- \frac{1}{5}x_0 + \frac{5}{21}x_2 - \frac{5}{9}x_4 + \frac{\pi}{4}x_5 - \frac{5}{11}x_6 = -\frac{1}{5}\\
&- \frac{1}{7}x_0 - \frac{1}{35}x_1 + \frac{1}{9}x_2 + \frac{1}{9}x_3 - \frac{1}{11}x_4 - \frac{5}{11}x_5 + \left(\frac{1}{13}+\frac{\pi}{4}\right)x_6 = -\frac{1}{9}
\end{aligned}
$$

The coefficients were changed to decimal fractions and considered only up to the fifth place after the decimal point. Six approximations were computed; for the first approximation $x_2 = x_3 = \ldots = x_6 = 0$ was used. The result is presented in table I. Insertion of equation (20a) in equation (3a), gives the torsion constant J_d as

$$J_d = 2R^4\left(-\frac{\pi}{8} + \frac{\pi}{4}x_0 - \frac{1}{3}x_1 + \frac{1}{3\times 5}x_3 - \frac{1}{5\times 7}x_5 + - \ldots\right) = \kappa R^4 \quad (25)$$

and the following approximations for κ computed exact to three places:

$\kappa_{(1)} = 0.414$ $\kappa_{(2)} = 0.326$ $\kappa_{(3)} = 0.300$

$\kappa_{(4)} = 0.298$ $\kappa_{(5)} = 0.300$ $\kappa_{(6)} = 0.298$

The third approximation computed from four linear equations already gives a torsion constant value that differs by no more than 2/3 percent from the rigorously computed value.

For the stress calculation, equation (20a) is inserted in equation (5a), so that

$$\tau_r = -\frac{M_d}{2\kappa R^3}\sum_1^n kx_k\lambda^{k-1}\sin k\varphi \quad (26)$$

$$\tau_\varphi = -\frac{M_d}{2\kappa R^3}\sum_1^n\left(-2\lambda + kx_k\lambda^{k-1}\cos k\varphi\right) \quad (27)$$

The shearing stresses $\tau_\varphi^{\varphi=\frac{\pi}{2}}$ and $\tau_r^{\varphi=\frac{\pi}{2}}$ at the straight boundary are

$$\tau_r^{\varphi=\frac{\pi}{2}} = -\frac{M_d}{2\kappa R^3}\left(x_1 - 3x_3\lambda^2 + 5x_5\lambda^4 - + \ldots\right) \quad (28)$$

$$\tau_\varphi^{\varphi=\frac{\pi}{2}} = -\frac{M_d}{2\kappa R^3}\left(-2\lambda - 2x_2\lambda + 4x_4\lambda^3 - 6x_6\lambda^5 + -\right) \tag{29}$$

For $\lambda = 1$ the shearing stresses are

$$\tau_r^{\lambda=1} = -\frac{M_d}{2\kappa R^3}\left(x_1 \sin \varphi + 2x_2 \sin 2\varphi + . . .\right) \tag{30}$$

$$\tau_\varphi^{\lambda=1} = -\frac{M_d}{2\kappa R^3}\left(-2 + x_1 \cos \varphi + 2x_2 \cos 2\varphi + . . .\right) \tag{31}$$

The two maximum shearing stresses are:

$$\tau_r^{\varphi=\frac{\pi}{2},\lambda=0} = \tau_{max} = -\frac{x_1}{2\kappa}\frac{M_d}{R^3} \tag{32}$$

$$\tau_\varphi^{\varphi=\pi,\lambda=1} = \tau_\varphi^{C} = -\frac{M_d}{2\kappa R^3}\left(-2 - x_1 + 2x_2 - 3x_3 + . . .\right) \tag{33}$$

Of these expressions $\tau_\varphi^{\varphi=\frac{\pi}{2}}$ and $\tau_1^{\lambda=1}$ must at least approximately disappear.

Now for a check of the extent to which these conditions are met for the different approximations and also of the extent of the differences between the approximated and the exact values of τ_{max} and τ_φ^{C}. The results are represented in table II and figures 3 to 5. The fourth approximation already gives a serviceable result, which is somewhat further improved by the fifth and sixth approximations.

Figures 3 and 4 show the approximations for $\tau_\varphi^{\varphi=\frac{\pi}{2}}$ and $\tau_r^{\lambda=1}$ which really should disappear. It indicates good agreement in the fifth and sixth approximations. Figure 5 shows the shearing stress $\tau_r^{\varphi=\frac{\pi}{2}}$ at the straight boundary plotted against $\lambda = \frac{r}{R}$. The fourth, fifth, and sixth approximations differ little from each other and from the accurate stress distribution designated by g. A marked departure occurs in the immediate vicinity of the corner (point B in fig. 2).

2. The Segment

(a) <u>Approximate solution by the method of least squares</u>.- Since the cross section is symmetrical to $\varphi = 0$ (fig. 6) the formula (20a) is applied to the stress function f. The boundary values are given by

$$\overline{f}_{AB} = \frac{R^2}{4}\left(-\frac{\cos^2\alpha}{\cos^2\varphi} + \sum_0^n x_k \frac{\cos^k\alpha}{\cos^k\varphi} \cos k\varphi\right) \quad 0 \leqq \varphi \leqq \alpha \ . \ . \ . \tag{34a}$$

$$\overline{f}_{BC} = \frac{R^2}{4}\left(-1 + \sum_0^n x_k \cos k\varphi\right) \quad \alpha \leqq \varphi \leqq \pi \tag{34b}$$

If ds_1 is an element of the straight boundary AB and ds_2 an element of the arc BC,

$$ds_1 = R \cos \alpha \frac{d\varphi}{\cos^2\varphi} \tag{35a}$$

$$ds_2 = R\, d\varphi \tag{35b}$$

The expression that is to be made a minimum by the choice of the coefficients x_k reads

$$\overline{J} = \int_0^\alpha \left(-\frac{\cos^2\alpha}{\cos^2\varphi} + \sum_0^n x_k \frac{\cos^k\alpha}{\cos^k\varphi} \cos k\varphi\right)^2 d\varphi \frac{\cos\alpha}{\cos^2\varphi}$$

$$+ \int_\alpha^\pi \left(-1 + \sum_0^n x_k \cos k\varphi\right)^2 d\varphi \tag{36}$$

From $\frac{\partial J}{\partial x_k} = 0$ follows a linear equation system with symmetrical matrix for x_k, which is to be written in the form of equation (23). The coefficients A_{kl} and the right sides are given by

$$A_{00} = \pi - \alpha + \sin \alpha \tag{37a}$$

$$A_{kk} = \cos^{2k+1}\alpha \int_0^\alpha \frac{\cos^2 k\varphi}{\cos^{2k+2}\varphi}\, d\varphi + \frac{\pi - \alpha}{2} - \frac{\sin 2k\alpha}{4k} \quad k = 1, 2, \ldots n \tag{37b}$$

$$A_{kl} = \cos^{k+l+1}\alpha \int_0^\alpha \frac{\cos k\varphi \cos l\varphi}{\cos^{k+l+2}\varphi}\, d\varphi - \frac{1}{2}\left[\frac{\sin (k - l)\varphi}{k - l} + \frac{\sin (k + l)\varphi}{k + l}\right] \begin{matrix} k = 1, 2, \ldots n \\ l = 0, 1, \ldots n \end{matrix} \tag{37c}$$

$$B_0 = \pi - \alpha + \sin \alpha \left(\cos^2\alpha + \frac{1}{3}\sin^2\alpha\right) \tag{37d}$$

$$B_l = \cos^{l+3}\alpha \int_0^\alpha \frac{\cos l\varphi}{\cos^{l+4}\varphi}\, d\varphi - \frac{\sin l\alpha}{l} \quad l = 1, 2, \ldots n \tag{37e}$$

The integrals appearing in equation (37) are of the form $\int_0^\alpha \frac{\cos p\varphi}{\cos^q\varphi}$; they can be defined by expressing $\cos p\varphi$ by $\cos^p\varphi$, $\cos^{p-2}\varphi$, etc. A reproduction of the somewhat elaborate formulas is omitted.

The matrix A_{kl} including the right-hand sides B_l of equation (23) were computed to five places with the calculating machine as functions of α. To keep the paper work within tolerable limits the process was carried to $k, l = 6$. The result is shown in figure 6. The unknowns $x_0, x_1 \ldots x_6$ were computed by equation (23), by the Gauss method. The result is given in table IV.

After $x_0, x_1 \ldots x_6$ have been determined the torsion constant J_d and the shearing stresses can be computed.

By equation (3a) the torsion constant is

$$J_d = 8 \iint_{ABC} f(r, \varphi) r \, dr \, d\varphi$$

The double integral is to be extended over the area ABC (fig. 6)

$$\bar{r} = R \frac{\cos \alpha}{\cos q}$$

$$\iint_{ABC} = \iint_{AOC} + \iint_{OBC} = \int_0^\alpha d\varphi \int_0^{\bar{r}} f(r, \varphi) r \, dr$$

$$+ \int_\alpha^\pi d\varphi \int_0^R f(r, \varphi) r \, dr$$

Insertion of the expression for the shearing function f from equation (23) in this formula gives

$$J_d = R^4 \left[-\frac{\pi - \alpha}{2} - \frac{\sin 2\alpha}{4} \left(\cos^2\alpha + \frac{1}{3} \sin^2\alpha \right) \right.$$

$$+ x_0 \left(\pi - \alpha + \frac{\sin 2\alpha}{2} \right)$$

$$\left. + 2 \sum_1^n \frac{x_k}{k + 2} \left(J_k - \frac{\sin k\alpha}{k} \right) \right] \equiv \kappa R^4 \qquad (38)$$

with

$$J_k = \cos^{k+2}\alpha \int_0^{\alpha} \frac{\cos k\varphi}{\cos^{k+2}\varphi}\, d\varphi \tag{39}$$

By equations (5) and (5a) the shearing stresses are

$$\tau_{xy} = \frac{M_d}{2\kappa R^3}\left[2\lambda \cos\varphi - \sum_1^n k\, x_k \lambda^{k-1} \cos(k-1)\varphi\right] \tag{40}$$

$$\tau_{xz} = -\frac{M_d}{2\kappa R^3}\left[2\lambda \cos\varphi + \sum_2^n k\, x_k \lambda^{k-1} \sin(k-1)\varphi\right] \tag{41}$$

$$\tau_{\varphi} = \frac{M_d}{2\kappa R^3}\left(2\lambda - \sum_1^n k\, x_k \lambda^{k-1} \cos k\varphi\right) \tag{42}$$

$$\tau_r = -\frac{M_d}{2\kappa R^3} \sum_1^n k\, x_k \lambda^{k-1} \sin k\varphi \tag{43}$$

Particularly important are the formulas for the stresses at the boundaries. These are
on AB

$$\tau_{xy}^{AB}\ \frac{M_d}{2\kappa R^3}\left[2\cos\alpha - \sum_1^n k\, x_k \left(\frac{\cos\alpha}{\cos\varphi}\right)^{k-1} \cos(k-1)\varphi\right] \tag{40a}$$

$$\tau_{xz}^{AB} = -\frac{M_d}{2\kappa R^3}\left[2\cos\alpha \tan\varphi\right.$$

$$\left. + \sum_1^n k\, x_k \left(\frac{\cos\alpha}{\cos\varphi}\right)^{k-1} \sin(k-1)\varphi\right] \tag{41a}$$

on BC

$$\tau_{\varphi}^{BC} = \frac{M_d}{2\kappa R^3}\left(2 - \sum_1^n k\ x_k \cos k\varphi\right) \qquad (42a)$$

$$\tau_{r}^{BC} = -\frac{M_d}{2\kappa R^3}\sum_1^n k\ x_k \sin k\varphi \qquad (43a)$$

Of these equations (41a) and (43a) must disappear (at least approximately).

Lastly there are the formulas for the shearing stresses in A and C $(\varphi = \pi)$.

$$\tau_{xy}^{A} = \tau_{max} = \frac{M_d}{2\kappa R^3}\left(2 \cos \alpha - \sum_1^n k\ x_k \cos^{k-1}\alpha\right) \qquad (40b)$$

$$\tau_{\varphi}^{C} = \frac{M_d}{2\kappa R^3}\left[2 + \sum_1^n (-1)^{k-1} k\ x_k\right] \qquad (42b)$$

The numerical values for the torsion constant and the particularly interesting shearing stresses τ_{xy}^{A} and τ_{φ}^{C} follow from equations (38), (40b), and (42b). These are also included in table IV and in figure 7 plotted against α.

(b) Solution formula by Fourier series.- The torsion problem for segmental cross section can also be solved by means of the Fourier series. The method is briefly explained.

To transform equation (1a) we put

$$f = -\frac{r^2}{4} + \Phi(r, \varphi) \qquad (44)$$

Φ must be a potential function which assumes the values

$$\overline{\Phi} = \frac{\overline{r^2}}{4} \qquad (45)$$

at the section boundary.

Therefore

$$\overline{\Phi} = \begin{cases} \dfrac{R^2}{4}\,\dfrac{\cos^2\alpha}{\cos^2\varphi} & 0 < \varphi \leqq \alpha \\[2ex] \dfrac{R^2}{4} & \alpha \leqq \varphi \leqq \pi \end{cases} \tag{46}$$

This even function of φ is developed in a Fourier series in the interval $-\pi \leqq \varphi \leqq +\pi$

$$\overline{\Phi} = \sum_0^\infty a_n \cos n\varphi \tag{47}$$

with

$$a_o = \frac{R^2}{4}\,\frac{1}{\pi}\left(\pi - \alpha + \frac{\sin 2\alpha}{2}\right)$$

$$a_n = \frac{R^2}{4}\,\frac{2}{\pi}\cos^2\alpha \int_0^\alpha \frac{\cos n\varphi}{\cos^2\varphi}\,\alpha\varphi \quad \frac{\sin n\alpha}{n} \tag{47a}$$

The potential function $\Phi(r, \varphi)$ is built up with the aid of yet to be determined coefficients from particular solutions.

$$\Phi = \sum_0^\infty b_n r^n \cos n\varphi \tag{48}$$

At the boundary Φ assumes the following values

$$\left.\begin{aligned} 0 \leqq \varphi \leqq \alpha \quad \overline{\Phi} &= \sum_0^\infty b_r R^n \frac{\cos^n\alpha}{\cos^n\varphi}\cos n\varphi \\ \alpha \leqq \varphi \leqq \pi \quad \overline{\Phi} &= \sum_0^\infty b_n R^n \cos n\varphi \end{aligned}\right\} \tag{49}$$

This even function of φ is also developed in interval $-\pi \leq \varphi \leq +\pi$ in a Fourier series

$$\overline{\Phi} = \sum_0^\infty B_n \cos n\varphi \tag{50}$$

$$\left.\begin{aligned}
B_o &= 2\left[\frac{1}{2\pi}\int_0^\alpha\left(\sum_0^\infty b_k R^k \frac{\cos^k\alpha}{\cos^k\varphi}\cos k\varphi\right)d\varphi\right. \\
&\qquad\left. + \frac{1}{2\pi}\int_\alpha^\pi\left(\sum_0^\infty l_k R^k \cos k\varphi\right)d\varphi\right] \\
B_n &= 2\left[\frac{1}{\pi}\int_0^\alpha\left(\sum_0^\infty b_k R^k \frac{\cos^k\alpha}{\cos^k\varphi}\cos k\varphi\right)\cos n\varphi\, d\varphi\right. \\
&\qquad\left. + \frac{1}{\pi}\int_0^\alpha\left(\sum_0^\infty b_k R^k \cos k\varphi\right)\cos n\varphi\, d\varphi\right]
\end{aligned}\right\} \tag{50a}$$

The Fourier series (equations (47) and (50)) obtained for the boundary values of Φ must be identical, that is

$$B_n = a_n \qquad n = 0, 1, \ldots \tag{51}$$

This is an infinite linear equation system for the looked-for coefficients b_n. With

$$b_n R^n = \frac{R^2}{4} x_n \tag{52}$$

the system reads

$$\pi x_0 + \sum_1^{\infty} a_{on} x_n = \pi - \alpha + \frac{\sin 2\alpha}{2}$$

$$\sum_1^{\infty} a_{kn} x_n = \cos^2\alpha \int_0^{\alpha} \frac{\cos k\varphi}{\cos^2\varphi} d\varphi \, \frac{\sin k\alpha}{k} \qquad (53)$$

The coefficients a_{on} and a_{kn} are given by

$$\left.\begin{aligned}
a_{on} &= \cos^n\alpha \int_0^{\alpha} \frac{\cos n\varphi}{\cos^n\varphi} d\varphi - \frac{\sin n\alpha}{n} \quad n = 1, 2, \ldots \\
a_{nn} &= \cos^n\alpha \int_0^{\alpha} \frac{\cos^2 n\varphi}{\cos^n\varphi} d\varphi + \frac{\pi - \alpha}{2} \frac{\sin 2n\alpha}{4n} \\
a_{kn} &= \cos^n\alpha \int_0^{\alpha} \frac{\cos k\varphi \cos n\varphi}{\cos^n\varphi} d\varphi \, \frac{\sin(n - k)\alpha}{2(n - k)} \, \frac{\sin(n + k)\alpha}{2(n + k)}
\end{aligned}\right\} \qquad (54)$$

Obviously $a_{kn} \neq a_{nk}$, that is, the matrix $\left(a_{kn}\right)$ is not symmetrical.

To solve for given α the torsion problem by this process the Fourier series must be limited to finitely many terms; in other words, the system (53) must be approximated by the section method. For example, going as far as x_7 inclusive means that $7^2 = 49$ factors a_{kn} have to be computed. The numerical calculation thus becomes very tedious and is therefore omitted.

III. CHECK OF THEORETICAL RESULT BY TEST

With the setup described in reference 3 the torsion constant J_d of a member of segmental section was optically determined; while the maximum shearing stress τ_{max} (point A in fig. 6) was determined by means of stress measurements. The shaft sketched in reference 3, figure 1 was machined to $d = 70$ millimeters and a flat surface milled out which gave the desired section. The milled surfaces corresponded to the angles $\alpha = 20^\circ$, 40°, 60°, and 80°. The comparison is illustrated in figure 7. The agreement is plainly sufficient.

Translated by J. Vanier
National Advisory Committee
for Aeronautics

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1. Trefftz, E.: Ein Gegenstück zum Ritzschen Verfahren. Verhandlungen des 2. internationalen Kongresses für technische Mechanik, Zürich 1926, p. 131.

2. Bergmann, St.: Ein Näherungsverfahren zur Lösung gewisser partieller, linearer Differentialgeeichungen. Z. angew. Math. u. Mech. Bd. 11 (1931), p. 323.

3. Weigand, A.: Ermittlung der Formziffer der auf Verdrehung beanspruchten abgesetzten Welle mit Hilfe von Feindehnungsmessungen. Luftf-Forschg. Bd. 20 (1943), Lfg. 7, p. 217. (Also available as NACA TM No. 1179.)

TABLE I

THE APPROXIMATIONS FOR THE UNKNOWN IN THE EQUATION SYSTEM (23) APPLICABLE TO THE SEMICIRCLE

x_0	x_1	x_2	x_3	x_4	x_5	x_6
First approximation						
0.486	-0.654	-------	--------	--------	--------	------
Second approximation						
0.1135	-1.399	-0.638	--------	--------	--------	------
Third approximation						
0.0136	-1.6553	-0.9412	0.3004	--------	--------	------
Fourth approximation						
-0.0086	-1.7364	-1.0859	-0.4566	-0.10095	--------	------
Fifth approximation						
-0.0068	-1.7310	-1.0711	-0.43465	-0.0835	-0.00855	------
Sixth approximation						
-0.00022	-1.6978	-0.9967	-0.3251	0.0308	0.0900	0.0359

TABLE II

APPROXIMATE VALUES FOR τ_{max} AND τ_{φ}^{C}

First approximation	Second approximation	Third approximation	Fourth approximation	Fifth approximation	Sixth approximation	Exact value
$\frac{R^3}{M_d}\tau_{max} = 0.790$	2.15	2.76	2.91	2.88	2.85	2.85
$\frac{R^3}{M_d}\tau_{\varphi}^{C} = 1.625$	2.79	2.2	2.46	2.47	2.41	2.44

TABLE IV

THE SOLUTIONS OF THE LINEAR EQUATION SYSTEM (23) INCLUDING THE TORSION CONSTANT AND THE SHEARING STRESSES τ_{xy}^{A} AND τ_{φ}^{C} AS FUNCTIONS OF α.

α^0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	$\kappa = \frac{J_d}{R^4}$	$\frac{R^3}{M_d}\tau_{xy}^{A}$	$\frac{R^3}{M_d}\tau_{\varphi}^{C}$
0	1	0	0	0	0	0	0	1.571	0.637	0.637
10	.9990	-.0023	-.0022	-.0022	-.0021	-.0021	-.0020	1.567	.642	.638
20	.9903	-.0191	-.0183	-.0171	-.0155	-.0136	-.0116	1.541	.694	.66
30	.9642	-.0697	-.0644	-.0562	-.0478	-.0347	-.0237	1.470	.794	.70
40	.9106	-.1715	-.1515	-.1219	-.0871	-.0544	-.0261	1.342	.91	.74
50	.8209	-.3375	-.2814	-.2046	-.1255	-.0609	-.0163	1.155	1.054	.83
60	.6867	-.5790	-.4511	-.2900	-.1443	-.0464	.0004	.933	1.24	.96
70	.5060	-.8865	-.6309	-.3355	-.1092	.0035	.0230	.706	1.52	1.19
80	.2743	-1.2704	-.8331	-.3711	-.0705	.0339	.0261	.479	2.03	1.65
90	-.0002	-1.6978	-.9967	-.3251	.0308	.0900	.0359	.298	2.85	2.41

TABLE III

THE FACTORS A_{kl} AND THE RIGHT-HAND SIDES B_l OF THE EQUATION

SYSTEM (23) AS FUNCTIONS OF α

α^0	A_{00}	A_{10}	A_{20}	A_{30}	A_{40}	A_{50}	A_{60}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}
0	$\pi =$ 3.14159	0	0	0	0	0	0	$\frac{\pi}{2}$	0	0	0	0
10	3.14071	-0.00264	-0.00434	-0.00597	-0.00749	-0.00887	-0.01010	1.56644	-0.00602	-0.00754	-0.00906	-0.01037
20	3.13455	-.02063	-.03272	-.04247	-.04924	-.05262	-.05251	1.53758	-.04408	-.05244	-.05774	-.05964
30	3.11799	-.06699	-.09968	-.11683	-.11651	-.10000	-.07143	1.46749	-.12799	-.13727	-.13006	-.10825
40	3.08625	-.15038	-.20373	-.20317	-.15391	-.07593	.00365	1.35273	-.24459	-.22345	-.16254	-.08115
50	3.03497	-.27364	-.32573	-.25217	-.11243	.04360	.11953	1.20477	-.35922	-.25841	-.11017	.02214
60	2.96042	-.43301	-.43301	-.21651	.04330	.17321	.12372	1.04720	-.43301	-.21651	.00000	.10825
70	2.85956	-.61830	-.48806	-.07954	.21147	.17907	-.03490	.90916	-.44352	-.12180	.08882	.10030
80	2.73014	-.81380	-.45969	.12798	.28926	.01578	-.19320	.81686	-.39819	-.03306	.10238	.03324

α^0	A_{16}	A_{22}	A_{23}	A_{24}	A_{25}	A_{26}	A_{33}	A_{34}	A_{35}	A_{36}	A_{44}	A_{45}
0	0	$\frac{\pi}{2}$	0	0	0	0	$\frac{\pi}{2}$	0	0	0	$\frac{\pi}{2}$	0
10	-0.01152	1.56316	-0.00915	-0.01054	-0.01179	-0.01278	1.65020	-0.01192	-0.01309	-0.01409	1.55763	-0.01427
20	-.05811	1.51724	-.06052	-.06446	-.06517	-.06266	1.50095	-.06905	-.06899	-.06607	1.49955	-.07096
30	-.07615	1.42574	-.14845	-.13793	-.11549	-.08488	1.42150	-.14005	-.12220	-.09866	1.37016	-.12379
40	-.00323	1.31838	-.23081	-.17387	-.10820	-.04464	1.35679	-.18916	-.15487	-.11441	1.36711	-.20866
50	.09043	1.23693	-.26244	-.16859	-.07796	-.01005	1.30337	-.26029	-.22327	-.15017	1.23249	-.35332
60	.08660	1.19875	-.28146	-.17939	-.07732	.00773	1.19875	-.39590	-.29383	-.10052	1.06112	-.44229
70	.00008	1.17772	-.34199	-.23029	-.05523	.08821	1.02084	-.51914	-.23139	.07240	.98608	-.41085
80	-.04833	1.11997	-.46668	-.24891	.07134	.18163	.84906	-.51685	-.07622	.16559	.99088	-.43332

α^0	A_{46}	A_{55}	A_{56}	A_{66}	B_0	B_1	B_2	B_3	B_4	B_5	B_6	
0	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	0	0	0	0	0	0	
10	-0.01518	1.55551	-0.01616	1.55387	3.13722	-0.00608	-0.00771	-0.00924	-0.01065	-0.01191	-0.01299	
20	-.06784	1.49977	-.06923	1.50187	3.10788	-.04569	-.05565	-.06285	-.06676	-.06714	-.06399	
30	-.11065	1.43861	-.13284	1.42295	3.03466	-.13916	-.15801	-.16013	-.14508	-.11547	-.07639	
40	-.17969	1.31970	-.25970	1.26975	2.90919	-.28602	-.29300	-.24914	-.16596	-.06679	.02124	
50	-.25712	1.16030	-.36888	1.18899	2.73528	-.46628	-.41439	-.26394	-.07524	.07570	.13652	
60	-.21033	1.10596	-.36623	1.12647	2.52745	-.65052	-.47631	-.17321	.09279	.18867	.10825	
70	-.13485	1.06098	-.45178	.95862	2.30638	-.80750	-.45506	-.00143	.23550	.15234	-.06247	
80	-.16127	.88529	-.53599	.92319	2.09340	-.92437	-.35538	.18899	.25968	-.02243	-.18709	

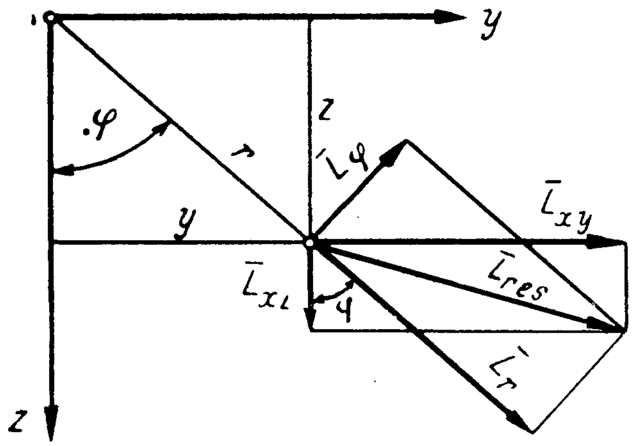

Figure 1.- The coordinates of the section points and the shearing stress components.

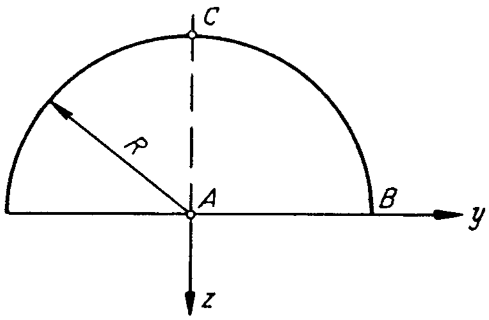

Figure 2.- Semicircular section with coordinate system.

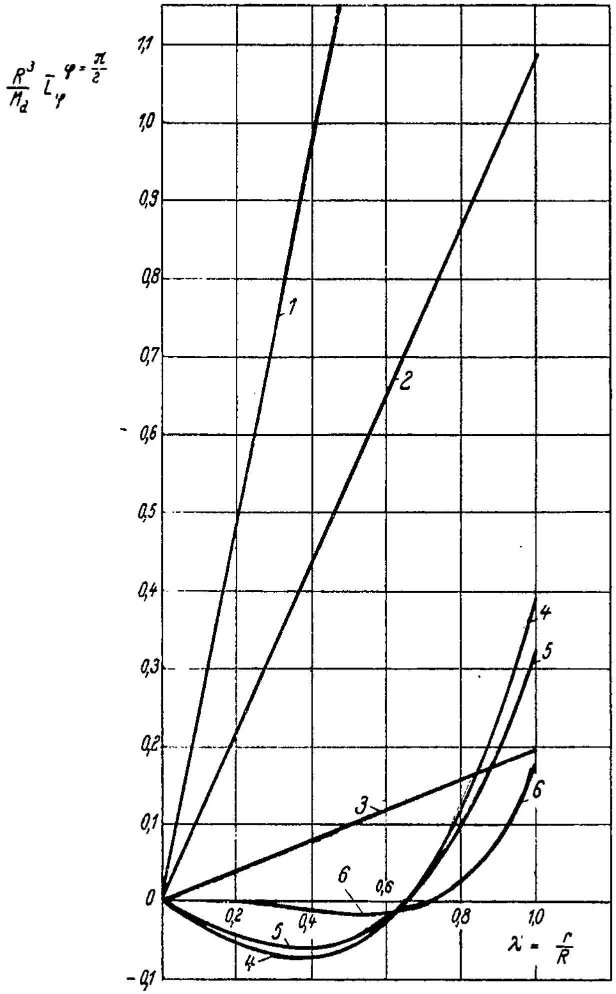

Figure 3.- Approximations for the shearing stress τ_{φ} at the straight boundary AB of the semicircular section.
1:1. Approximation 4:4. Approximation
2:2. Approximation 5:5. Approximation
3:3. Approximation 6:6. Approximation

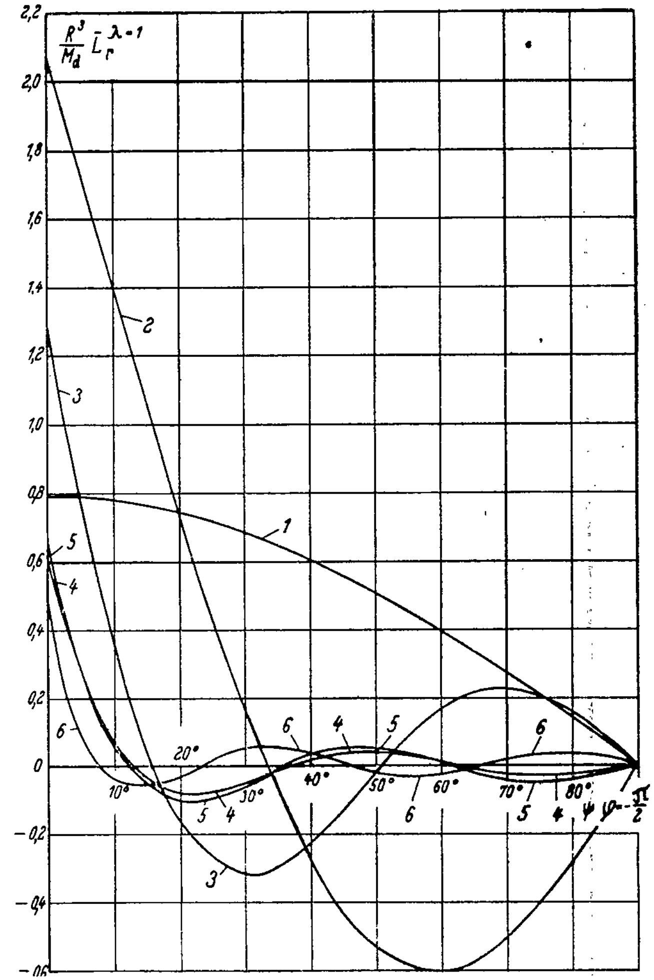

Figure 4.- The approximations for τ_r at the boundary BC of the semicircular section.
1:1. Approximation 4:4. Approximation
2:2. Approximation 5:5. Approximation
3:3. Approximation 6:6. Approximation

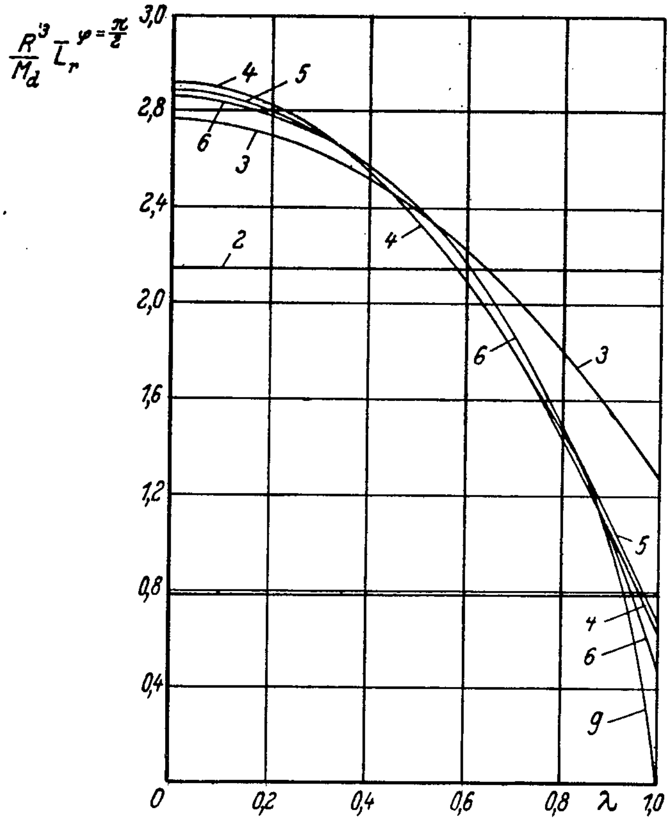

Figure 5.- The approximations for τ_r at the boundary AB of the semicircular section.

1:1. Approximation 4:4. Approximation
2:2. Approximation 5:5. Approximation
3:3. Approximation 6:6. Approximation
g: exact solution

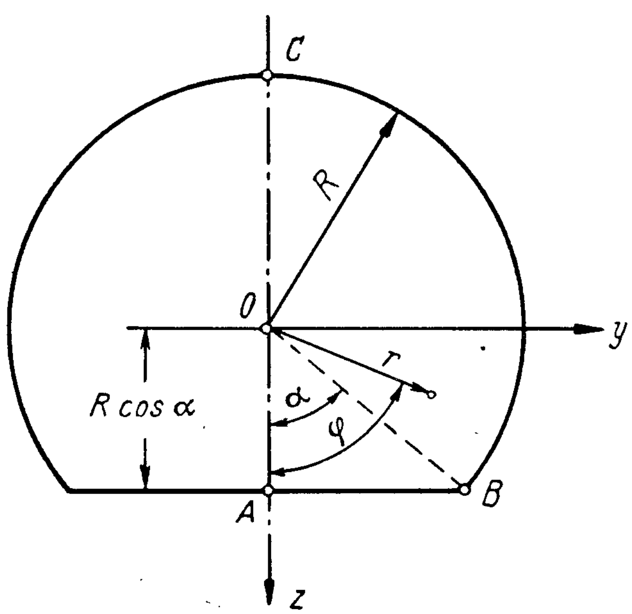

Figure 6.- Notation at segment.

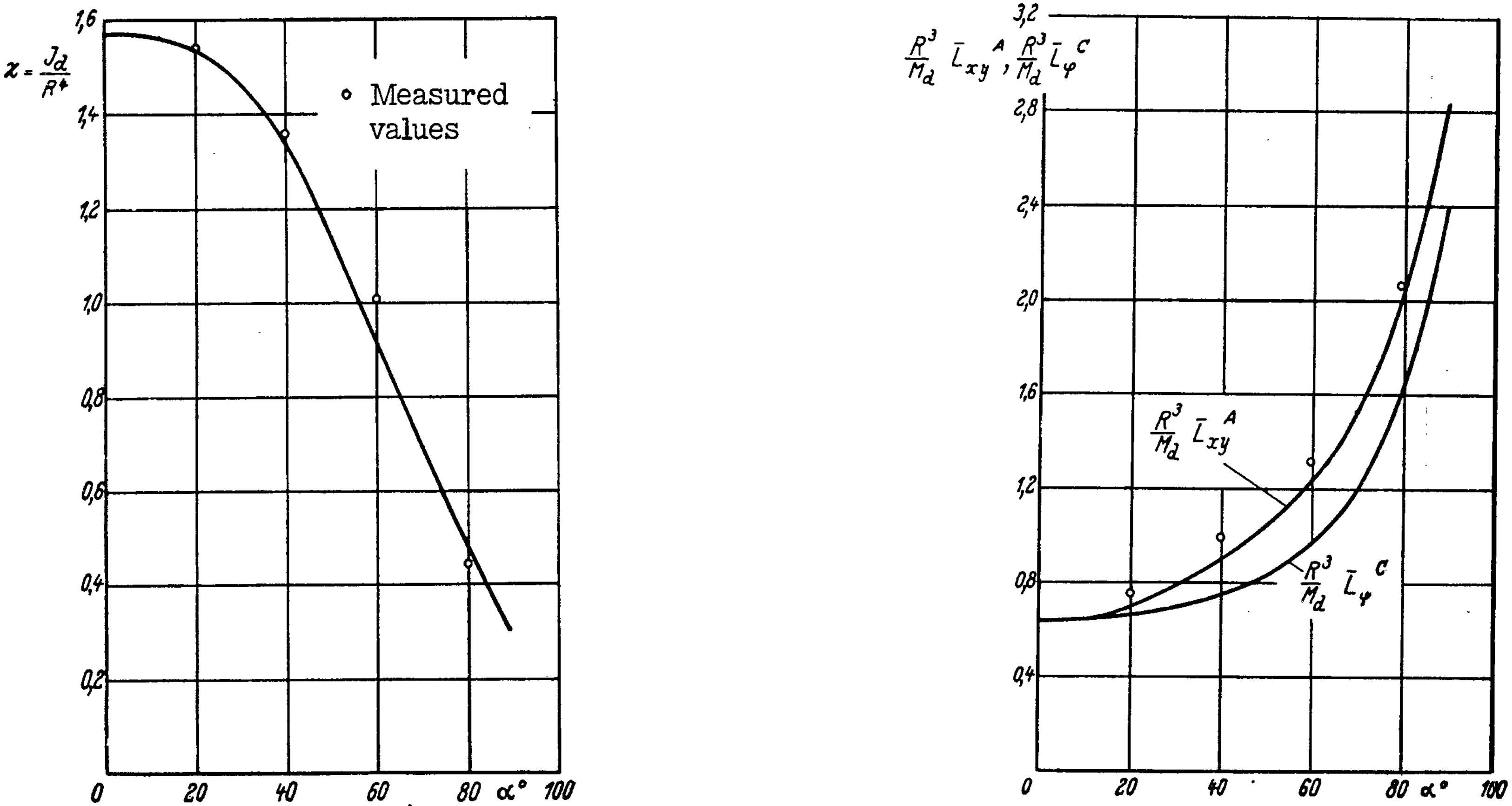

Figure 7.- The torsion constant J_d and the shearing stresses τ_φ^C and $\tau_{xy}^A = \tau_{max}$ of the segmental section plotted against the angle at center α; comparison between theoretical values (full curve) and experimental values.